ON A CLASS OF h-FOURIER INTEGRAL OPERATORS

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Abstract. In this paper, we study the L^2-boundedness and L^2-compactness of a class of h-Fourier integral operators. These operators are bounded (respectively compact) if the weight of the amplitude is bounded (respectively tends to 0).

1. Introduction

For $\varphi \in \mathcal{S}(\mathbb{R}^n)$ (the Schwartz space), the integral operators

$$F_h\varphi(x) = \iint e^{\frac{i}{h}(S(x,\theta)-y\theta)} a(x,\theta)\,\varphi(y)\,dyd\theta \tag{1.1}$$

appear naturally in the expression of the solutions of the semiclassical hyperbolic partial differential equations and in the expression of the C^∞-solution of the associate Cauchy's problem. Which appear two C^∞-functions, the phase function $\phi(x,y,\theta) = S(x,\theta) - y\theta$ and the amplitude a..

Since 1970, many efforts have been made by several authors in order to study these type of operators (see, e.g.,[2, 7, 8, 5, 9]). The first works on Fourier integral operators deal with local properties. On the other hand, K. Asada and D. Fujiwara ([2]) have studied for the first time a class of Fourier integral operators defined on $\mathbb{R}^n$.

For the h-Fourier integral operators, an interesting question is under which conditions on a and S these operators are bounded on L^2 or are compact on L^2.

It has been proved in [2] by a very elaborated proof and with some hypothesis on the phase function ϕ and the amplitude a that all operators of the form:

$$(I(a,\phi)\varphi)(x) = \iint\limits_{\mathbb{R}^n_y\times\mathbb{R}^N_\theta} e^{i\phi(x,\theta,y)} a(x,\theta,y)\,\varphi(y)\,dyd\theta \tag{1.2}$$

are bounded on L^2 where, $x \in \mathbb{R}^n$, $n \in \mathbb{N}^*$ and $N \in \mathbb{N}$ (if $N = 0$, θ doesn't appear in (1.2)). The technique used there is based on the fact that the operators $I(a,\phi)I^*(a,\phi), I^*(a,\phi)I(a,\phi)$ are pseudodifferential and it uses Caldéron-Vaillancourt's theorem (here $I(a,\phi)^*$ is the adjoint of $I(a,\phi)$).

In this work, we apply the same technique of [2] to establish the boundedness and the compactness of the operators (1.1). To this end we give a brief and simple proof for a result of [2] in our framework.

We mainly prove the continuity of the operator F_h on $L^2(\mathbb{R}^n)$ when the weight of the amplitude a is bounded. Moreover, F_h is compact on $L^2(\mathbb{R}^n)$ if this weight tends

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to zero. Using the estimate given in [12, 13] for h-pseudodifferential (h-admissible) operators, we also establish an L^2-estimate of $\|F_h\|$.

We note that if the amplitude a is juste bounded, the Fourier integral operator F is not necessarily bounded on $L^2(\mathbb{R}^n)$. Recently, M. Hasanov [7] and we [1] constructed a class of unbounded Fourier integral operators with an amplitude in the Hörmander's class $S^0_{1,1}$ and in $\bigcap\limits_{0<\rho<1} S^0_{\rho,1}$.

To our knowledge, this work constitutes a first attempt to diagonalize the h-Fourier integral operators on $L^2(\mathbb{R}^n)$ (relying on the compactness of these operators).

2. A general class of h-Fourier integral operators

If $\varphi \in \mathcal{S}(\mathbb{R}^n)$, we consider the following integral transformations

$$(I(a,\phi;h)\varphi)(x) = \iint\limits_{\mathbb{R}^n_y\times\mathbb{R}^N_\theta} e^{\frac{i}{h}\phi(x,\theta,y)}a(x,\theta,y)\varphi(y)dy\,d\theta \tag{2.3}$$

where, $x \in \mathbb{R}^n$, $n \in \mathbb{N}^*$ and $N \in \mathbb{N}$ (if $N = 0$, θ doesn't appear in (2.3)).

In general the integral (2.3) is not absolutely convergent, so we use the technique of the oscillatory integral developed by Hörmander. The phase function ϕ and the amplitude a are assumed to satisfy the following hypothesis:

(H1) $\phi \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}^N_\theta \times \mathbb{R}^n_y, \mathbb{R})$ (ϕ is a real function)

(H2) For all $(\alpha,\beta,\gamma) \in \mathbb{N}^n \times \mathbb{N}^N \times \mathbb{N}^n$, there exists $C_{\alpha,\beta,\gamma} > 0$

$$|\partial_y^\gamma \partial_\theta^\beta \partial_x^\alpha \phi(x,\theta,y)| \le C_{\alpha,\beta,\gamma}\lambda^{(2-|\alpha|-|\beta|-|\gamma|)_+}(x,\theta,y)$$

where $\lambda(x,\theta,y) = (1+|x|^2+|\theta|^2+|y|^2)^{1/2}$ called the weight and

$$(2-|\alpha|-|\beta|-|\gamma|)_+ = \max(2-|\alpha|-|\beta|-|\gamma|, 0)$$

(H3) There exist $K_1, K_2 > 0$ such that $\forall(x,\theta,y) \in \mathbb{R}^n_x \times \mathbb{R}^N_\theta \times \mathbb{R}^n_y$

$$K_1\lambda(x,\theta,y) \le \lambda(\partial_y\phi, \partial_\theta\phi, y) \le K_2\lambda(x,\theta,y)$$

(H3*) There exist $K_1^*, K_2^* > 0$ such that, $\forall(x,\theta,y) \in \mathbb{R}^n_x \times \mathbb{R}^N_\theta \times \mathbb{R}^n_y$

$$K_1^*\lambda(x,\theta,y) \le \lambda(x, \partial_\theta\phi, \partial_x\phi) \le K_2^*\lambda(x,\theta,y)$$

For any open subset Ω of $\mathbb{R}^n_x \times \mathbb{R}^N_\theta \times \mathbb{R}^n_y$, $\mu \in \mathbb{R}$ and $\rho \in [0,1]$, we set

$$\Gamma^\mu_\rho(\Omega) = \big\{a \in C^\infty(\Omega) : \forall(\alpha,\beta,\gamma) \in \mathbb{N}^n \times \mathbb{N}^N \times \mathbb{N}^n,\ \exists C_{\alpha,\beta,\gamma} > 0 :$$
$$|\partial_y^\gamma \partial_\theta^\beta \partial_x^\alpha a(x,\theta,y)| \le C_{\alpha,\beta,\gamma}\lambda^{\mu-\rho(|\alpha|+|\beta|+|\gamma|)}(x,\theta,y)\big\}$$

When $\Omega = \mathbb{R}^n_x \times \mathbb{R}^N_\theta \times \mathbb{R}^n_y$, we denote $\Gamma^\mu_\rho(\Omega) = \Gamma^\mu_\rho$.

To give a meaning to the right hand side of (2.3), we consider $g \in \mathcal{S}(\mathbb{R}^n_x \times \mathbb{R}^N_\theta \times \mathbb{R}^n_y)$, $g(0) = 1$. If $a \in \Gamma^\mu_0$, we define

$$a_\sigma(x,\theta,y) = g(x/\sigma, \theta/\sigma, y/\sigma)a(x,\theta,y), \quad \sigma > 0.$$

Theorem 2.1. *If ϕ satisfies (H1), (H2), (H3) and (H3*), and if $a \in \Gamma^\mu_0$, then*

1. For all $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $\lim_{\sigma\to+\infty}[I(a_\sigma,\phi;h)\varphi](x)$ exists for every point $x \in \mathbb{R}^n$ and is independent of the choice of the function g. We define

$$(I(a,\phi;h)\varphi)(x) := \lim_{\sigma\to+\infty}(I(a_\sigma,\phi;h)\varphi)(x)$$

2. $I(a,\phi;h) \in \mathcal{L}(\mathcal{S}(\mathbb{R}^n))$ and $I(a,\phi;h) \in \mathcal{L}(\mathcal{S}'(\mathbb{R}^n))$ (here $\mathcal{L}(E)$ is the space of bounded linear mapping from E to E and $\mathcal{S}'(\mathbb{R}^n)$ the space of all distributions with temperate growth on $\mathbb{R}^n$).

Proof. see [8] or [12, propostion II.2]. □

Example 2.2. *Let's give two examples of operators of the form* (2.3) *which satisfy* $(H1)$ *to* $(H3)^*$*:*

(1) *The Fourier transform* $\mathcal{F}\psi\,(x) = \int\limits_{\mathbb{R}^n} e^{-ixy}\psi\,(y)\,dy,\ \psi \in \mathcal{S}(\mathbb{R}^n)$,

(2) *Pseudodifferential operators* $A\psi\,(x) = (2\pi)^{-n} \int\limits_{\mathbb{R}^{2n}} e^{i(x-y)\theta} a\,(x,y,\theta)\,\psi\,(y)\,dyd\theta$, $\psi \in \mathcal{S}(\mathbb{R}^n),\ a \in \Gamma_0^\mu\left(\mathbb{R}^{3n}\right)$.

3. Assumptions and Preliminaries

We consider the special form of the phase function

$$\phi(x,y,\theta) = S(x,\theta) - y\theta \tag{3.4}$$

where S satisfies

(G1) $S \in C^\infty(\mathbb{R}^n_x \times \mathbb{R}^n_\theta, \mathbb{R})$,

(G2) For each $(\alpha,\beta) \in \mathbb{N}^{2n}$, there exist $C_{\alpha,\beta} > 0$, such that

$$|\partial_x^\alpha \partial_\theta^\beta S(x,\theta)| \leq C_{\alpha,\beta}\lambda(x,\theta)^{(2-|\alpha|-|\beta|)},$$

(G3) There exists $\delta_0 > 0$ such that

$$\inf_{x,\theta\in\mathbb{R}^n} |\det \frac{\partial^2 S}{\partial x \partial\theta}(x,\theta)| \geq \delta_0.$$

Lemma 3.1 ([11])**.** *Let's assume that S satisfies (G1), (G2), (G3). Then the function $\phi(x,y,\theta) = S(x,\theta) - y\theta$ satisfies (H1), (H2), (H3) and (H3*).*

Lemma 3.2 ([11])**.** *If S satisfies (G1), (G2) and (G3), then there exists $C_2 > 0$ such that for all $(x,\theta),(x',\theta') \in \mathbb{R}^{2n}$,*

$$|x-x'| + |\theta-\theta'| \leq C_2\left[|(\partial_\theta S)(x,\theta) - (\partial_\theta S)(x',\theta')| + |\theta-\theta'|\right] \tag{3.5}$$

when $\theta = \theta'$ in (3.5), there exists $C_2 > 0$, such that for all $(x,x',\theta) \in \mathbb{R}^{3n}$,

$$|x-x'| \leq C_2|(\partial_\theta S)(x,\theta) - (\partial_\theta S)(x',\theta)|. \tag{3.6}$$

Proposition 3.3. *If S satisfies (G1) and (G2), then there exists a constant $\epsilon_0 > 0$ such that the phase function ϕ given in* (3.4) *belongs to $\Gamma_1^2(\Omega_{\phi,\epsilon_0})$ where*

$$\Omega_{\phi,\epsilon_0} = \left\{(x,\theta,y) \in \mathbb{R}^{3n};\ \ |\partial_\theta S(x,\theta) - y|^2 < \epsilon_0\left(|x|^2 + |y|^2 + |\theta|^2\right)\right\}.$$

Proof. We have to show that: $\exists\varepsilon_0 > 0,\ \forall\alpha,\beta,\gamma \in \mathbb{N}^n, \exists C_{\alpha,\beta,\gamma} > 0$;

$$\left|\partial_x^\alpha \partial_\theta^\beta \partial_y^\gamma \phi(x,\theta,y)\right| \leq C_{\alpha,\beta,\gamma}\lambda(x,\theta,y)^{(2-|\alpha|-|\beta|-|\gamma|)},\ \forall(x,\theta,y) \in \Omega_{\phi,\varepsilon_0}. \tag{3.7}$$

- If $|\gamma| = 1$, then $\left|\partial_x^\alpha \partial_\theta^\beta \partial_y^\gamma \phi(x,\theta,y)\right| = \left|\partial_x^\alpha \partial_\theta^\beta(-\theta)\right| = \left\{\begin{array}{ll} 0 & \text{if } |\alpha| \neq 0 \\ \left|\partial_\theta^\beta(-\theta)\right| & \text{if } \alpha = 0 \end{array}\right.$;
- If $|\gamma| > 1$, then $\left|\partial_x^\alpha \partial_\theta^\beta \partial_y^\gamma \phi(x,\theta,y)\right| = 0$.

Hence the estimate (3.7) is satisfied.

If $|\gamma| = 0$, then $\forall \alpha, \beta \in \mathbb{N}^n$; $|\alpha| + |\beta| \leq 2$, $\exists C_{\alpha,\beta} > 0$;

$$\left|\partial_x^\alpha \partial_\theta^\beta \phi(x,\theta,y)\right| = \left|\partial_x^\alpha \partial_\theta^\beta S(x,\theta) - \partial_x^\alpha \partial_\theta^\beta (y\theta)\right| \leq C_{\alpha,\beta} \lambda(x,\theta,y)^{(2-|\alpha|-|\beta|)}.$$

If $|\alpha| + |\beta| > 2$, one has $\partial_x^\alpha \partial_\theta^\beta \phi(x,\theta,y) = \partial_x^\alpha \partial_\theta^\beta S(x,\theta)$. In $\Omega_{\phi,\varepsilon_0}$ we have

$$|y| = |\partial_\theta S(x,\theta) - y - \partial_\theta S(x,\theta)| \leq \sqrt{\varepsilon_0}\left(|x|^2 + |y|^2 + |\theta|^2\right)^{1/2} + C_3 \lambda(x,\theta),\ C_3 > 0.$$

For ε_0 sufficiently small, we obtain a constant $C_4 > 0$ such that

$$|y| \leq C_4 \lambda(x,\theta),\ \forall (x,\theta,y) \in \Omega_{\phi,\varepsilon_0}. \tag{3.8}$$

This inequality leads to the equivalence

$$\lambda(x,\theta,y) \simeq \lambda(x,\theta) \text{ in } \Omega_{\phi,\varepsilon_0} \tag{3.9}$$

thus the assumption $(G2)$ and (3.9) give the estimate (3.7). □

Using (3.9), we have the following result.

Proposition 3.4. *If $(x,\theta) \to a(x,\theta)$ belongs to $\Gamma_k^m(\mathbb{R}_x^n \times \mathbb{R}_\theta^n)$, then $(x,\theta,y) \to a(x,\theta)$ belongs to $\Gamma_k^m(\mathbb{R}_x^n \times \mathbb{R}_\theta^n \times \mathbb{R}_y^n) \cap \Gamma_k^m(\Omega_{\phi,\epsilon_0})$, $k \in \{0,1\}$.*

4. L^2-boundedness and L^2-compactness of F_h

Theorem 4.1. *Let F_h be the integral operator of distribution kernel*

$$K(x,y;h) = \int_{\mathbb{R}^n} e^{\frac{i}{h}(S(x,\theta)-y\theta)} a(x,\theta)\widehat{d_h\theta} \tag{4.10}$$

where $\widehat{d_h\theta} = (2\pi h)^{-n} d\theta$, $a \in \Gamma_k^m(\mathbb{R}^{2n}_{x,\theta})$, $k = 0,1$ and S satisfies (G1), (G2) and (G3). Then $F_h F_h^*$ and $F_h^* F_h$ are h-pseudodifferential operators with symbol in $\Gamma_k^{2m}(\mathbb{R}^{2n})$, $k = 0,1$, given by

$$\sigma(F_h F_h^*)(x, \partial_x S(x,\theta)) \equiv |a(x,\theta)|^2 |(\det \frac{\partial^2 S}{\partial\theta\partial x})^{-1}(x,\theta)|$$
$$\sigma(F_h^* F_h)(\partial_\theta S(x,\theta),\theta) \equiv |a(x,\theta)|^2 |(\det \frac{\partial^2 S}{\partial\theta\partial x})^{-1}(x,\theta)|$$

we denote here $a \equiv b$ for $a, b \in \Gamma_k^{2p}(\mathbb{R}^{2n})$ if $(a-b) \in \Gamma_k^{2p-2}(\mathbb{R}^{2n})$ and σ stands for the symbol.

Proof. For all $v \in \mathcal{S}(\mathbb{R}^n)$, we have:

$$(F_h F_h^* v)(x) = \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} e^{\frac{i}{h}(S(x,\theta)-S(\widetilde{x},\theta))} a(x,\theta)\overline{a}(\widetilde{x},\theta) d\widetilde{x}\widehat{d\theta}. \tag{4.11}$$

The main idea to show that $F_h F_h^*$ is a h- pseudodifferential operator, is to use the fact that $(S(x,\theta) - S(\widetilde{x},\theta))$ can be expressed by the scalar product $\langle x - \widetilde{x}, \xi(x,\widetilde{x},\theta)\rangle$ after considering the change of variables $(x,\widetilde{x},\theta) \to (x,\widetilde{x},\xi = \xi(x,\widetilde{x},\theta))$. The distribution kernel of $F_h F_h^*$ is

$$K(x,\tilde{x};h) = \int_{\mathbb{R}^n} e^{\frac{i}{h}(S(x,\theta)-S(\tilde{x},\theta))} a(x,\theta)\overline{a}(\tilde{x},\theta)\widehat{d_h\theta}.$$

We obtain from (3.6) that if

$$|x - \widetilde{x}| \geq \frac{\varepsilon}{2}\lambda(x,\widetilde{x},\theta) \text{ (where } \varepsilon > 0 \text{ is sufficiently small)}$$

then

$$|(\partial_\theta S)(x,\theta)-(\partial_\theta S)(\widetilde{x},\theta)| \geq \frac{\varepsilon}{2C_2}\lambda(x,\widetilde{x},\theta). \tag{4.12}$$

Choosing $\omega \in C^\infty(\mathbb{R})$ such that

$$\omega(x) \geq 0, \quad \forall x \in \mathbb{R}$$
$$\omega(x) = 1 \quad \text{if } x \in [-\frac{1}{2},\frac{1}{2}]$$
$$\operatorname{supp}\omega \subset]-1,1[$$

and setting

$$b(x,\tilde{x},\theta) := a(x,\theta)\overline{a}(\tilde{x},\theta) = b_{1,\epsilon}(x,\tilde{x},\theta) + b_{2,\epsilon}(x,\tilde{x},\theta)$$
$$b_{1,\epsilon}(x,\tilde{x},\theta) = \omega(\frac{|x-\tilde{x}|}{\epsilon\lambda(x,\tilde{x},\theta)})b(x,\tilde{x},\theta)$$
$$b_{2,\epsilon}(x,\tilde{x},\theta) = [1-\omega(\frac{|x-\tilde{x}|}{\epsilon\lambda(x,\tilde{x},\theta)})]b(x,\tilde{x},\theta).$$

We have $K(x,\widetilde{x};h) = K_{1,\epsilon}(x,\widetilde{x};h) + K_{2,\epsilon}(x,\widetilde{x};h)$, where

$$K_{j,\epsilon}(x,\tilde{x};h) = \int_{\mathbb{R}^n} e^{\frac{i}{h}(S(x,\theta)-S(\tilde{x},\theta))} b_{j,\epsilon}(x,\tilde{x},\theta)\widehat{d_h\theta}, \quad j=1,2.$$

We will study separately the kernels $K_{1,\epsilon}$ and $K_{2,\epsilon}$. □

Proof. For all h, we have

$$K_{2,\epsilon}(x,\widetilde{x};h) \in \mathcal{S}(\mathbb{R}^n \times \mathbb{R}^n).$$

Indeed, using the oscillatory integral method, there is a linear partial differential operator L of order 1 such that

$$L\left(e^{\frac{i}{h}(S(x,\theta)-S(\tilde{x},\theta))}\right) = e^{\frac{i}{h}(S(x,\theta)-S(\tilde{x},\theta))}$$

where $L = -ih\left|(\partial_\theta S)(x,\theta)-(\partial_\theta S)(\widetilde{x},\theta)\right|^{-2}\sum_{l=1}^{n}\left[(\partial_{\theta_l}S)(x,\theta)-(\partial_{\theta_l}S)(\widetilde{x},\theta)\right]\partial_{\theta_l}.$

The transpose operator of L is

$${}^tL = \sum_{l=1}^{n} F_l(x,\widetilde{x},\theta;h)\,\partial_{\theta_l} + G(x,\widetilde{x},\theta;h)$$

where $F_l(x,\widetilde{x},\theta) \in \Gamma_0^{-1}(\Omega_\varepsilon)$, $G(x,\widetilde{x},\theta) \in \Gamma_0^{-2}(\Omega_\varepsilon)$

$$\begin{cases} F_l(x,\widetilde{x},\theta;h) = ih\left|(\partial_\theta S)(x,\theta)-(\partial_\theta S)(\widetilde{x},\theta)\right|^{-2}\left((\partial_{\theta_l}S)(x,\theta)-(\partial_{\theta_l}S)(\widetilde{x},\theta)\right) \\ G(x,\widetilde{x},\theta;h) = ih\sum\limits_{l=1}^{n}\partial_{\theta_l}\left[\left|(\partial_\theta S)(x,\theta)-(\partial_\theta S)(\widetilde{x},\theta)\right|^{-2}\left((\partial_{\theta_l}S)(x,\theta)-(\partial_{\theta_l}S)(\widetilde{x},\theta)\right)\right] \\ \Omega_\varepsilon = \left\{(x,\tilde{x},\theta)\in\mathbb{R}^{3n};\ |\partial_\theta S(x,\theta)-\partial_\theta S(\tilde{x},\theta)| > \frac{\varepsilon}{2C_2}\lambda(x,\tilde{x},\theta)\right\}. \end{cases}$$

On the other hand we prove by induction on q that

$$\left({}^tL\right)^q b_{2,\varepsilon}(x,\tilde{x},\theta) = \sum_{\substack{|\gamma|\leq q\\ \gamma\in\mathbb{N}^n}} g_{\gamma,q}(x,\tilde{x},\theta)\,\partial_\theta^\gamma b_{2,\varepsilon}(x,\tilde{x},\theta),\ g_\gamma^{(q)} \in \Gamma_0^{-q}(\Omega_\varepsilon),$$

and so,

$$K_{2,\varepsilon}(x,\tilde{x}) = \int\limits_{\mathbb{R}^n} e^{\frac{i}{h}(S(x,\theta)-S(\tilde{x},\theta))} \left({}^tL\right)^q b_{2,\varepsilon}(x,\tilde{x},\theta)\,\widehat{d\theta}.$$

Using Leibnitz's formula, $(G2)$ and the form $\left({}^tL\right)^q$, we can choose q large enough such that

$$\forall \alpha,\alpha',\beta,\beta' \in \mathbb{N}^n, \exists C_{\alpha,\alpha',\beta,\beta'} > 0, \quad \sup_{x,\widetilde{x}\in\mathbb{R}^n} \left| x^{\alpha}\widetilde{x}^{\alpha'}\partial_x^{\beta}\partial_{\widetilde{x}}^{\beta'} K_{2,\varepsilon}(x,\widetilde{x};h)\right| \le C_{\alpha,\alpha',\beta,\beta'}.$$

Next, we study K_1^{ϵ}: this is more difficult and depends on the choice of the parameter ϵ. It follows from Taylor's formula that

$$S(x,\theta) - S(\widetilde{x},\theta) = \langle x-\widetilde{x}, \xi(x,\widetilde{x},\theta)\rangle_{\mathbb{R}^n},$$

$$\xi(x,\widetilde{x},\theta) = \int_0^1 (\partial_x S)(\widetilde{x}+t(x-\widetilde{x}),\theta)dt.$$

We define the vectorial function

$$\widetilde{\xi}_{\epsilon}(x,\widetilde{x},\theta) = \omega\big(\frac{|x-\tilde{x}|}{2\epsilon\lambda(x,\tilde{x},\theta)}\big)\xi(x,\widetilde{x},\theta) + \big(1-\omega(\frac{|x-\tilde{x}|}{2\epsilon\lambda(x,\tilde{x},\theta)})\big)(\partial_x S)(\widetilde{x},\theta).$$

We have

$$\widetilde{\xi}_{\varepsilon}(x,\widetilde{x},\theta) = \xi(x,\widetilde{x},\theta) \ \text{ on } \ \operatorname{supp} b_{1,\varepsilon}.$$

Moreover, for ε sufficiently small,

$$\lambda(x,\theta) \simeq \lambda(\widetilde{x},\theta) \simeq \lambda(x,\widetilde{x},\theta) \ \text{ on } \ \operatorname{supp} b_{1,\varepsilon}. \tag{4.13}$$

Let us consider the mapping

$$\mathbb{R}^{3n} \ni (x,\widetilde{x},\theta) \to \Big(x,\widetilde{x},\widetilde{\xi}_{\varepsilon}(x,\widetilde{x},\theta)\Big) \tag{4.14}$$

for which Jacobian matrix is

$$\begin{pmatrix} I_n & 0 & 0 \\ 0 & I_n & 0 \\ \partial_x\widetilde{\xi}_{\varepsilon} & \partial_{\widetilde{x}}\widetilde{\xi}_{\varepsilon} & \partial_{\theta}\widetilde{\xi}_{\varepsilon} \end{pmatrix}.$$

We have

$$\frac{\partial\widetilde{\xi}_{\varepsilon,j}}{\partial\theta_i}(x,\widetilde{x},\theta) = \frac{\partial^2 S}{\partial\theta_i\partial x_j}(\widetilde{x},\theta) + \omega\left(\frac{|x-\tilde{x}|}{2\varepsilon\lambda(x,\tilde{x},\theta)}\right)\left(\frac{\partial\xi_j}{\partial\theta_i}(x,\widetilde{x},\theta) - \frac{\partial^2 S}{\partial\theta_i\partial x_j}(\widetilde{x},\theta)\right)$$
$$-\frac{|x-\tilde{x}|}{2\varepsilon\lambda(x,\tilde{x},\theta)}\frac{\partial\lambda}{\partial\theta_i}(x,\tilde{x},\theta)\,\lambda^{-1}(x,\tilde{x},\theta)\,\omega'\left(\frac{|x-\tilde{x}|}{2\varepsilon\lambda(x,\tilde{x},\theta)}\right)\left(\xi_j(x,\widetilde{x},\theta) - \frac{\partial S}{\partial x_j}(\widetilde{x},\theta)\right).$$

Thus, we obtain

$$\left|\frac{\partial\widetilde{\xi}_{\varepsilon,j}}{\partial\theta_i}(x,\widetilde{x},\theta) - \frac{\partial^2 S}{\partial\theta_i\partial x_j}(\widetilde{x},\theta)\right| \le \left|\omega\left(\frac{|x-\tilde{x}|}{2\varepsilon\lambda(x,\tilde{x},\theta)}\right)\right|\left|\frac{\partial\xi_j}{\partial\theta_i}(x,\widetilde{x},\theta) - \frac{\partial^2 S}{\partial\theta_i\partial x_j}(\widetilde{x},\theta)\right| +$$
$$\lambda^{-1}(x,\tilde{x},\theta)\left|\omega'\left(\frac{|x-\tilde{x}|}{2\varepsilon\lambda(x,\tilde{x},\theta)}\right)\right|\left|\xi_j(x,\widetilde{x},\theta) - \frac{\partial S}{\partial x_j}(\widetilde{x},\theta)\right|.$$

Now it follows from $(G2)$, (4.13) and Taylor's formula that

$$\begin{aligned} \left|\frac{\partial\xi_j}{\partial\theta_i}(x,\widetilde{x},\theta) - \frac{\partial^2 S}{\partial\theta_i\partial x_j}(\widetilde{x},\theta)\right| &\le \int\limits_0^1 \left|\frac{\partial^2 S}{\partial\theta_i\partial x_j}(\widetilde{x}+t(x-\widetilde{x}),\theta) - \frac{\partial^2 S}{\partial\theta_i\partial x_j}(\widetilde{x},\theta)\right| dt \\ &\le C_5\,|x-\widetilde{x}|\,\lambda^{-1}(x,\tilde{x},\theta),\ C_5 > 0 \qquad (4.15) \end{aligned}$$

$$\begin{aligned}\left|\xi_j(x,\widetilde{x},\theta)-\frac{\partial S}{\partial x_j}(\widetilde{x},\theta)\right| &\leq \int_0^1\left|\frac{\partial S}{\partial x_j}(\widetilde{x}+t(x-\widetilde{x}),\theta)-\frac{\partial S}{\partial x_j}(\widetilde{x},\theta)\right|dt\\ &\leq C_6|x-\widetilde{x}|,\ C_6>0\ . \qquad (4.16)\end{aligned}$$

From (4.15) and (4.16), there exists a positive constant $C_7>0$, such that

$$\left|\frac{\partial\widetilde{\xi}_{\varepsilon,j}}{\partial\theta_i}(x,\widetilde{x},\theta)-\frac{\partial^2 S}{\partial\theta_i\partial x_j}(\widetilde{x},\theta)\right|\leq C_7\varepsilon,\ \forall i,j\in\{1,...,n\}. \qquad (4.17)$$

If $\varepsilon<\frac{\delta_0}{2\widetilde{C}}$, then (4.17) and $(G3)$ yields the estimate

$$\delta_0/2\leq-\widetilde{C}\varepsilon+\delta_0\leq-\widetilde{C}\varepsilon+\det\frac{\partial^2 S}{\partial x\partial\theta}(x,\theta)\leq\det\partial_\theta\widetilde{\xi}_\varepsilon(x,\widetilde{x},\theta),\ \text{with }\widetilde{C}>0. \qquad (4.18)$$

If ε is such that (4.13) and (4.18) are true, then the mapping given in (4.14) is a global diffeomorphism of $\mathbb{R}^{3n}$. Hence there exists a mapping

$$\theta:\mathbb{R}^n\times\mathbb{R}^n\times\mathbb{R}^n\ni(x,\widetilde{x},\xi)\to\theta(x,\widetilde{x},\xi)\in\mathbb{R}^n$$

such that

$$\begin{cases}\widetilde{\xi}_\varepsilon(x,\widetilde{x},\theta(x,\widetilde{x},\xi))= & \xi\\ \theta\left(x,\widetilde{x},\widetilde{\xi}_\varepsilon(x,\widetilde{x},\theta)\right)= & x\\ \partial^\alpha\theta(x,\widetilde{x},\xi)=\mathcal{O}(1), & \forall\alpha\in\mathbb{N}^{3n}\setminus\{0\}\end{cases} \qquad (4.19)$$

If we change the variable ξ by $\theta(x,\widetilde{x},\xi)$ in $K_{1,\varepsilon}(x,\widetilde{x})$, we obtain:

$$K_{1,\varepsilon}(x,\widetilde{x})=\int_{\mathbb{R}^n}e^{i<x-\tilde{x},\xi>}b_{1,\varepsilon}(x,\tilde{x},\theta(x,\widetilde{x},\xi))\left|\det\frac{\partial\theta}{\partial\xi}(x,\widetilde{x},\xi)\right|\widehat{d\xi}. \qquad (4.20)$$

From (4.19) we have, for $k=0,1$, that $b_{1,\varepsilon}(x,\tilde{x},\theta(x,\widetilde{x},\xi))\left|\det\frac{\partial\theta}{\partial\xi}(x,\widetilde{x},\xi)\right|$ belongs to $\Gamma_k^{2m}\left(\mathbb{R}^{3n}\right)$ if $a\in\Gamma_k^m\left(\mathbb{R}^{2n}\right)$.

Applying the stationary phase theorem (c.f. [12],[13]) to (4.20), we obtain the expression of the symbol of the h-pseudodifferential operator $F_hF_h^*$:

$$\sigma(F_hF_h^*)=b_{1,\varepsilon}(x,\tilde{x},\theta(x,\widetilde{x},\xi))\left|\det\frac{\partial\theta}{\partial\xi}(x,\widetilde{x},\xi)\right|_{|\widetilde{x}=x}+R(x,\xi;h)$$

where $R(x,\xi;h)$ belongs to $\Gamma_k^{2m-2}\left(\mathbb{R}^{2n}\right)$ if $a\in\Gamma_k^m\left(\mathbb{R}^{2n}\right)$, $k=0,1$.

For $\tilde{x}=x$, we have $b_{1,\varepsilon}(x,\tilde{x},\theta(x,\widetilde{x},\xi))=|a(x,\theta(x,x,\xi))|^2$ where $\theta(x,x,\xi)$ is the inverse of the mapping $\theta\to\partial_x S(x,\theta)=\xi$. Thus

$$\sigma(F_hF_h^*)(x,\partial_x S(x,\theta))\equiv|a(x,\theta)|^2\left|\det\frac{\partial^2 S}{\partial\theta\partial x}(x,\theta)\right|^{-1}.$$

such that

$$\widetilde{\xi}_\epsilon(x,\widetilde{x},\theta(x,\widetilde{x},\xi))=\xi$$
$$\theta(x,\widetilde{x},\widetilde{\xi}_\epsilon(x,\widetilde{x},\theta))=x$$
$$\partial^\alpha\theta(x,\widetilde{x},\xi)=\mathcal{O}(1),\quad\forall\alpha\in\mathbb{N}^{3n}\setminus\{0\}$$

If we change the variable ξ by $\theta(x,\widetilde{x},\xi)$ in $K_{1,\epsilon}(x,\widetilde{x})$, we obtain

$$K_{1,\epsilon}(x,\widetilde{x})=\int_{\mathbb{R}^n}e^{i\langle x-\tilde{x},\xi\rangle}b_{1,\epsilon}(x,\tilde{x},\theta(x,\widetilde{x},\xi))\big|\det\frac{\partial\theta}{\partial\xi}(x,\widetilde{x},\xi)\big|\widehat{d\xi}.$$

Applying the stationary phase theorem, we obtain the expression of the symbol of the h-pseudodifferential operator $F_h F_h^*$, is

$$\sigma(F_h F_h^*)(x, \partial_x S(x,\theta)) \equiv |a(x,\theta)|^2 \Big| \det \frac{\partial^2 S}{\partial\theta\partial x}(x,\theta) \Big|^{-1}.$$

The distribution kernel of the integral operator $\mathcal{F}(F_h^* F_h)\mathcal{F}^{-1}$ is

$$\widetilde{K}(\theta, \widetilde{\theta}) = \int_{\mathbb{R}^n} e^{-\frac{i}{h}\left(S(x,\theta)-S\left(x,\tilde{\theta}\right)\right)} \, \overline{a}(x,\theta) \; a\left(x, \tilde{\theta}\right) \widehat{dx}.$$

Remark that we can deduce $\widetilde{K}(\theta, \widetilde{\theta})$ from $K(x, \widetilde{x})$ by replacing x by θ. On the other hand, all assumptions used here are symmetrical on x and θ, therefore $\mathcal{F}(F^*F)\mathcal{F}^{-1}$ is a nice h-pseudodifferential operator with symbol

$$\sigma(\mathcal{F}(F_h^* F_h)\mathcal{F}^{-1})\,(\theta, -\partial_\theta S(x,\theta)) \equiv |a(x,\theta)|^2 \left| \det \frac{\partial^2 S}{\partial x \partial\theta}(x,\theta) \right|^{-1}.$$

Thus the symbol of F^*F is given by (c.f. [10])

$$\sigma(F_h^* F_h)(\partial_\theta S(x,\theta), \theta) \equiv |a(x,\theta)|^2 \left| \det \frac{\partial^2 S}{\partial x \partial\theta}(x,\theta) \right|^{-1}.$$

□

Corollary 4.2. *Let F_h be the integral operator with the distribution kernel*

$$K(x,y;h) = \int_{\mathbb{R}^n} e^{\frac{i}{h}(S(x,\theta)-y\theta)} a(x,\theta) \widehat{d_h\theta}$$

where $a \in \Gamma_0^m(\mathbb{R}^{2n}_{x,\theta})$ and S satisfies (G1), (G2) and (G3). Then, we have:

1. *For any m such that $m \le 0$, F_h can be extended as a bounded linear mapping on $L^2(\mathbb{R}^n)$*
2. *For any m such that $m < 0$, F_h can be extended as a compact operator on $L^2(\mathbb{R}^n)$.*

Proof. It follows from theorem 4.1 that $F_h^* F_h$ is a h-pseudodifferential operator with symbol in $\Gamma_0^{2m}\left(\mathbb{R}^{2n}\right)$.

1) If $m \le 0$, the weight $\lambda^{2m}(x,\theta)$ is bounded, so we can apply the Caldéron-Vaillancourt theorem (see [4, 12, 13]) for $F_h^* F_h$ and obtain the existence of a positive constant $\gamma(n)$ and a integer $k\,(n)$ such that

$$\|(F_h^* F_h)\, u\|_{L^2(\mathbb{R}^n)} \le \gamma(n)\; Q_{k(n)}\,(\sigma(F_h^* F_h))\, \|u\|_{L^2(\mathbb{R}^n)}\,,\ \forall u \in \mathcal{S}(\mathbb{R}^n)$$

where

$$Q_{k(n)}\,(\sigma(F_h^* F_h)) = \sum_{|\alpha|+|\beta| \le k(n)} \sup_{(x,\theta)\in\mathbb{R}^{2n}} \left| \partial_x^\alpha \partial_\theta^\beta \sigma(F_h^* F_h)(\partial_\theta S(x,\theta), \theta) \right|$$

Hence, we have $\forall u \in \mathcal{S}(\mathbb{R}^n)$

$$\|F_h u\|_{L^2(\mathbb{R}^n)} \le \|F_h^* F_h\|^{1/2}_{\mathcal{L}(L^2(\mathbb{R}^n))} \|u\|_{L^2(\mathbb{R}^n)} \le \left(\gamma(n)\; Q_{k(n)}\,(\sigma(F_h^* F_h))\right)^{1/2} \|u\|_{L^2(\mathbb{R}^n)}\,.$$

Thus F_h is also a bounded linear operator on $L^2(\mathbb{R}^n)$.

2) If $m < 0$, $\lim\limits_{|x|+|\theta|\to+\infty} \lambda^m(x,\theta) = 0$, and the compactness theorem (see [12, 13]) show that the operator $F_h^* F_h$ can be extended as a compact operator on $L^2(\mathbb{R}^n)$.

Thus, the Fourier integral operator F_h is compact on $L^2(\mathbb{R}^n)$. Indeed, let $(\varphi_j)_{j\in\mathbb{N}}$ be an orthonormal basis of $L^2(\mathbb{R}^n)$, then

$$\left\| F_h^* F_h - \sum_{j=1}^{n} < \varphi_j, . > F_h^* F_h \varphi_j \right\| \underset{n\to+\infty}{\longrightarrow} 0.$$

Since F_h is bounded, we have $\forall \psi \in L^2(\mathbb{R}^n)$

$$\left\| F_h \psi - \sum_{j=1}^{n} < \varphi_j, \psi > F_h \varphi_j \right\|^2 \leq$$

$$\left\| F_h^* F_h \psi - \sum_{j=1}^{n} < \varphi_j, \psi > F_h^* F_h \varphi_j \right\| \left\| \psi - \sum_{j=1}^{n} < \varphi_j, \psi > \varphi_j \right\|$$

then

$$\left\| F_h - \sum_{j=1}^{n} < \varphi_j, . > F_h \varphi_j \right\| \underset{n\to+\infty}{\longrightarrow} 0$$

□

Example 4.3. *We consider the function given by*

$$S(x,\theta) = \sum_{\substack{|\alpha|+|\beta|=2 \\ \alpha,\beta\in\mathbb{N}^n}} C_{\alpha,\beta} x^\alpha \theta^\beta, \ \textit{for} \ (x,\theta) \in \mathbb{R}^{2n}$$

where $C_{\alpha,\beta}$ are real constants. This function satisfies $(G1)$, $(G2)$ and $(G3)$.

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